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On the normal modes of parallel flow of inviscid stratified fluid

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The overall pattern of normal modes of parallel flow of inviscid stratified fluid is examined. For a given flow and wavenumber the modes are divided into five classes, some of which may be empty: (i) a finite class of non-singular unstable modes; (ii) a conjugate finite class of non-singular damped stable modes; (iii) a finite class of singular stable modes, each of these having a branch point and being the limit of unstable modes; (iv) a discrete class of modified internal gravity waves, these being non-singular stable modes (if the density decreases with height everywhere); (v) a continuous class of singular stable modes. The modified internal gravity waves are described asymptotically for large values of the Richardson number. These asymptotic results are related to and extended by numerical calculations for a sinusoidal basic velocity profile and a Bickley jet. The wave speeds for small values of the Richardson number are found to depend only upon the local behaviour of the mean flow near an overall simple maximum or minimum of the velocity profile. Finally some difficulties in the use of the Howard formula for perturbation at a curve of marginal stability are elucidated.

1. Introduction

Of the scores of papers on the linear instability of plane parallel flow of a stratified fluid, none seems to have given more than an occasional stable mode and none seems to have treated the overall pattern of the modes (see the surveys of Drazin & Howard 1966; Howard & Maslowe 1973). Yet it is well known that there is an infinity of stable modes, that they are essential to solve initial-value problems, and that all modes are stable when the local Richardson number is nowhere less than one quarter. Further, stable modes in the absence of a basic flow, namely internal gravity waves, have been studied extensively (see the books of Yih 1965; Turner 1973), and are recognized as important in many applications, notably to meteorology and oceanography. Yet in the phenomena to which the theory of internal gravity waves is applied there is frequently substantial shear in the mean flow. This shear seems to have been recognized in work on forced oscillations, that is to say on propagation of internal gravity waves from a source, but not in work on the closely related free oscillations, that is to say on the normal modes.

All these problems are governed by what is called the Taylor–Goldstein† equation, namely

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi+JN^2\phi/(U-c) = 0, \tag{1}$$

where $U(z)$ is the dimensionless velocity of the basic flow in the horizontal x direction, $N(z)$ is the dimensionless local Brunt–Väisälä frequency, J is the overall Richardson number, primes denote differentiations with respect to the dimensionless height z, and the stream function of the normal mode of disturbance is taken as $\phi(z)\exp\{i\alpha(x-ct)\}$. The complex eigenvalue c and eigenfunction ϕ are further determined by the boundary conditions

$$\phi = 0 \quad \text{at} \quad z = z_1, z_2, \tag{2}$$

which represent the vanishing of the vertical velocity on rigid horizontal walls. One may also consider semi-bounded flows by taking $z_2 = \infty$ or unbounded flows by taking $z_1 = -\infty$ and $z_2 = \infty$. For given values of the wavenumber α one seeks the eigensolution (c, ϕ) and hence the stability characteristics of dynamically similar flows specified by J, $U(z)$, $N(z)$, z_1 and z_2. A given mode is stable if and only if $\alpha c_i \leqslant 0$, where $c = c_r + ic_i$. Moreover it is said to be damped if $\alpha c_i < 0$, neutrally stable if $\alpha c_i = 0$ and marginally stable if in addition to $\alpha c_i = 0$ there exist unstable modes at neighbouring values of the wavenumber and Richardson number.

Here we examine all the modes, stable as well as unstable, seeking their overall features by numerical and analytic means. It is well known that the unstable modes, for each given pair of values of the wavenumber and Richardson number, are finite (possibly zero) in number, and may arise from the continuous spectrum as the Richardson number decreases. The literature suggests that the stable modes may be divided into four distinct classes for each pair of values of the wavenumber and Richardson number: (i) those that are conjugate to the unstable modes, (ii) those that are marginally stable, (iii) those that are essentially internal gravity waves modified by the basic shear and (iv) those that are essentially inertial modes modified by buoyancy. The members of the first class are in one–one correspondence with the unstable modes and decay exponentially in time. The second class is finite and has eigenfunctions with branch points at the critical layers, where $U(z) = c$. The third class is discrete and has real eigenvalues c outside the range of the velocity distribution $U(z)$ (providing the density decreases with height everywhere). The fourth class is continuous, has real eigenvalues within the range of $U(z)$ and has eigenfunctions discontinuous at the critical layers (Eliassen, Høiland & Riis 1953; Case 1960).

This overall pattern emerges from previous work on the unstable modes and on the continuous spectrum of stable modes. We analyse the overall pattern in § 2. In § 3 we analyse the discrete spectrum of stable modes at large values of the Richardson number, giving asymptotic formulae for the values of c. These results are confirmed and extended down to smallish values of the Richardson number

† This name was coined by Drazin (1958) in honour of the work of Taylor (1931) and Goldstein (1931) but in ignorance of the independent derivation of the equation by Haurwitz (1931).

in § 4. Further asymptotic results, for small values of the Richardson number, are given in § 5. These agree with the numerical results and fill in the picture of the discrete spectrum of stable modes. The unstable modes for a sinusoidal profile are used in § 6 to illuminate the applicability of Howard's formula for perturbation of marginally stable modes.

2. General theory

Although the essence of our methods below is quite widely applicable, it is more intelligible if we consider those flows for which we have

$$N^2 = 1, \quad z_1 = -\pi, \quad z_2 = \pi. \tag{3}$$

We shall treat such flows principally but shall remark upon other density distributions and upon unbounded flows.

First consider the behaviour in the limit as $J \to \infty$, when the Taylor–Goldstein equation becomes

$$L\phi \equiv \phi'' + (\gamma^{-2} - \alpha^2)\,\phi = 0, \tag{4}$$

where we define $\gamma \equiv c/J^{\frac{1}{2}}$. This gives the eigensolutions

$$c = \pm J^{\frac{1}{2}}\gamma_0 \equiv \pm J^{\frac{1}{2}}/(\alpha^2 + \tfrac{1}{4}n^2)^{\frac{1}{2}}, \quad \phi = \phi_0 \equiv \sin \tfrac{1}{2}n(\pi + z) \quad \text{for} \quad n = 1, 2, \ldots. \tag{5}$$

This limit may be regarded as the limit of either infinite buoyancy or zero dimensional scale of the basic velocity. It gives the well-known discrete spectrum (5) of internal gravity waves (or a continuous spectrum if the channel is of infinite height).

To express the problem in terms of a bounded operator so that standard results of spectral theory may be used, we next rewrite the Taylor–Goldstein equation without approximation in the form

$$L\phi = J^{-\frac{1}{2}} S\phi, \tag{6}$$

where

$$S(z) \equiv J^{\frac{1}{2}}\{\gamma^{-2} - (\gamma - U/J^{\frac{1}{2}})^{-2} + J^{-\frac{1}{2}}U''/(U/J^{\frac{1}{2}} - \gamma)\}. \tag{7}$$

After some routine analysis (cf. Friedman 1956, chap. 3) it follows that

$$\phi(z') = J^{-\frac{1}{2}} \int_{-\pi}^{\pi} G(z, z')\, S(z)\, \phi(z)\, dz, \tag{8}$$

where the Green's function is defined by

$$G(z, z') \equiv -\frac{1}{\mu \sin 2\mu\pi} \begin{Bmatrix} \sin \mu(\pi - z) \sin \mu(\pi + z') & \text{for} \quad z' \leqslant z \leqslant \pi \\ \sin \mu(\pi - z') \sin \mu(\pi + z) & \text{for} \quad -\pi \leqslant z \leqslant z' \end{Bmatrix} \tag{9}$$

and μ by

$$\mu \equiv (\gamma^{-2} - \alpha^2)^{\frac{1}{2}}, \tag{10}$$

provided that $\sin 2\mu\pi \neq 0$. Equation (8) is expressed in terms of an integral and thus a bounded operator (unless c lies in the range of $U(z)$ or μ is half an integer).

Now we may follow Dikiy & Katayev (1971), who applied spectral theory to a similar problem of barotropic instability. Applying their arguments to (8), and those surveyed by Drazin & Howard (1966) for the unstable modes and for the initial-value problem, we deduce the following picture. In the limit as $J \to \infty$ for fixed α and n, each mode is neutrally stable, belonging to the countable infinity of solutions (5). Some modes, however, may remain unstable as $J \to \infty$ if α or n is not fixed (see for example Huppert 1973). For finite values of J there may also be a continuous spectrum of neutrally stable eigensolutions with c in the range of $U(z)$ (see for example the survey by Drazin & Howard (1966) of the work of Eliassen *et al.* (1953) on plane Couette flow). Solution of the initial-value problem by Laplace transforms seems to imply that the normal modes are both complete and independent† and that therefore when the continuous spectrum arises the discrete spectrum is decreased. This pattern of stable modes continues as J decreases to zero with α fixed, although the discrete modes may decrease in number as the continuous spectrum grows. However, some discrete unstable modes may arise from the continuous spectrum as J decreases so far that $J/U'^2 < \frac{1}{4}$ somewhere. The marginal modes at the onset of instability are singular neutral modes with a branch point at the critical layer $z = z_c$, where $U(z_c) = c$. Finally, when $J = 0$, there is a finite number (possibly zero) of unstable modes, usually one for each point of inflexion of the basic velocity profile (Howard 1964), and the continuous spectrum of stable modes. The discrete spectrum of stable modes must finally vanish in the limit as $J \to 0$ but may remain for any positive value of J, however small (see §§ 4 and 5). It should be noted also that the unstable modes are in one–one correspondence with conjugate damped modes.

3. The internal gravity waves for large J

The integral form (8) of the problem implies (cf. Friedman 1956, p. 228) also that the expansions

$$\gamma = \gamma_0 + J^{-\frac{1}{2}}\gamma_1 + J^{-1}\gamma_2 + \ldots, \quad \phi = \phi_0 + J^{-\frac{1}{2}}\phi_1 + \ldots \tag{11}$$

for each integer n converge for sufficiently large values of J. We have already found γ_0 and ϕ_0 [equation (5)]. To find γ_1, ϕ_1, γ_2, etc. it seems easiest to revert to a differential equation. So we now rewrite the Taylor–Goldstein equation without approximation in the form

$$L_n\phi \equiv \phi'' + \tfrac{1}{4}n^2\phi = \{\gamma_0^{-2} - (\gamma - U/J^{\frac{1}{2}})^{-2} - J^{-\frac{1}{2}}U''/(\gamma - U/J^{\frac{1}{2}})\}\,\phi, \tag{12}$$

expand in powers of $J^{-\frac{1}{2}}$ and equate coefficients of $J^{-\frac{1}{2}}$, J^{-1}, etc.

The coefficients of $J^{-\frac{1}{2}}$ in (12) and the boundary conditions give

$$\left.\begin{aligned} L_n\phi_1 &= \{2\gamma_0^{-3}(\gamma_1 - U) - \gamma_0^{-1}U''\}\,\phi_0, \\ \phi_1 &= 0 \quad \text{at} \quad z = \pm\pi. \end{aligned}\right\} \tag{13}$$

But by integration by parts and use of the boundary conditions we have

$$\int_{-\pi}^{\pi} \{\phi_0 L_n\phi_1 - \phi_1 L_n\phi_0\}\,dz = 0. \tag{14}$$

† Note that when there is no continuous spectrum the normal modes for given values of J and α will comprise *two* complete sets of eigenfunctions, say $\phi_n^{(1)}(z)$ and $\phi_n^{(2)}(z)$, corresponding to two eigenvalues, say $c_n^{(1)}$ and $c_n^{(2)}$, in order to represent arbitrary initial perturbations of both the stream function and the density distribution. For the special case $J = \infty$, equation (5) gives $c_n^{(2)} = -c_n^{(1)}$ and $\phi_n^{(2)} \equiv \phi_n^{(1)}$ for all α, although these simple relations do not hold in general.

This identity and the equations for ϕ_0 and ϕ_1 now give the solubility condition for ϕ_1,

$$\int_{-\pi}^{\pi} \{2\gamma_0^{-3}(\gamma_1 - U) - \gamma_0^{-1}U''\}\phi_0^2\,dz = 0, \tag{15}$$

and thence

$$\gamma_1 = \pi^{-1}\int_{-\pi}^{\pi} (U + \tfrac{1}{2}\gamma_0^2 U'')\sin^2[\tfrac{1}{2}n(\pi+z)]\,dz. \tag{16}$$

To proceed further, let us take the particular basic velocity

$$U = \sin z \quad \text{for} \quad -\pi \leqslant z \leqslant \pi. \tag{17}$$

Then (16) and (17) give

$$\gamma_1 = 0. \tag{18}$$

Thence (13) becomes

$$L_n\phi_1 = \gamma_0^{-1}(1-2\gamma_0^{-2})\sin z \sin \tfrac{1}{2}n(\pi+z), \tag{19}$$

and with a little calculus we deduce that

$$\phi_1 = \left\{\begin{array}{l} -\tfrac{1}{2}\gamma_0^{-1}(1-2\gamma_0^{-2})\,(z\cos\tfrac{1}{2}z + \tfrac{1}{3}\sin\tfrac{3}{2}z) \quad \text{if} \quad n = 1, \\ \tfrac{1}{2}\gamma_0^{-1}(1-2\gamma_0^{-2})\,\{(n-1)^{-1}\cos[(\tfrac{1}{2}n-1)z + \tfrac{1}{2}n\pi] \\ \quad +(n+1)^{-1}\cos[(\tfrac{1}{2}n+1)z+\tfrac{1}{2}n\pi]\} \quad \text{if} \quad n \neq 1. \end{array}\right\} \tag{20}$$

Next, coefficients of J^{-1} give

$$\left.\begin{array}{c} L_n\phi_2 = (2\gamma_0\gamma_2 - 3U^2 + \gamma_0^2 U^2)\,\phi_0/\gamma_0^4 + \gamma_0^{-1}(1-2\gamma_0^{-2})\,U\phi_1, \\ \phi_2 = 0 \quad \text{at} \quad z = \pm\pi, \end{array}\right\} \tag{21}$$

and the solubility condition for ϕ_2 thence gives

$$\gamma_2 = \tfrac{1}{8}\gamma_0(2+\delta_{n2})\,(3\gamma_0^{-2}-1) + \tfrac{1}{4}\gamma_0(2\gamma_0^{-2}-1)^2/(1-n^2+\delta_{n1}), \tag{22}$$

where $\delta_{nm} = 0$ if $m \neq n$ and $\delta_{nm} = 1$ if $m = n$.

Note that we have taken the limit as $J \to \infty$ for fixed n (and α). If, however, n increases sufficiently rapidly with J we anticipate that the nth mode may cease to exist in order that there be a finite number of modes for each value of J, however large (provided that α is fixed).

Similar ideas to those in this section may be applied when N^2 varies with z, but one needs to know the appropriate solution ϕ_0 explicitly in order to invert the appropriately modified operator L_n. One may also apply the ideas to unbounded flow provided that there are uniform densities at infinity, i.e. that $N^2 \to 0$ as $z \to \pm\infty$. (However, if $N^2 \to 1$ as $z \to \pm\infty$, for example, the expansion (11) of ϕ in powers of $J^{-\frac{1}{2}}$ is not uniformly valid as $z \to \pm\infty$ and more refined methods are necessary.)

We briefly substantiate these extensions below, considering the instability of unbounded flow with variable N^2 such that $N^2 \to 0$ as $z \to \pm\infty$. Then the required generalization of (12) can be seen to be

$$L_n\phi \equiv \phi'' + (N^2/\gamma_0^2 - \alpha^2)\,\phi = \{N^2/\gamma_0^2 - N^2/(\gamma - U/J^{\frac{1}{2}})^2 - J^{-\frac{1}{2}}U''/(\gamma - U/J^{\frac{1}{2}})\}\,\phi.$$

Using expansions (11), we deduce that

$$L_n\phi_0 = 0; \quad \phi_0 \to 0 \quad \text{as} \quad z \to \pm\infty. \tag{23}$$

For a general given function $N^2(z)$, we know ϕ_0 and γ_0 only in principle, but can proceed to infer that

$$\psi_0(z) \equiv \phi_0(z) \int_0^z \phi_0^{-2}(z')\,dz'$$

is also a solution of the equation $L_n \psi_0 = 0$, although ψ_0 does not vanish as $z \to \pm\infty$. We proceed, much as we did before, to solve the system

$$\left.\begin{aligned} L_n \phi_1 &= \{2\gamma_0^{-3} N^2(\gamma_1 - U) - \gamma_0^{-1} U''\}\,\phi_0, \\ \phi_1 &\to 0 \quad \text{as} \quad z \to \pm\infty, \end{aligned}\right\} \tag{24}$$

instead of (13). Applying the self-adjoint condition (14) with the range of integration from $z = -\infty$ to $z = \infty$, we find

$$\gamma_1 = \int_{-\infty}^{\infty} (N^2 U + \tfrac{1}{2}\gamma_0^2 U'')\,\phi_0^2 dz \bigg/ \int_{-\infty}^{\infty} N^2 \phi_0^2\,dz \tag{25}$$

instead of (16). One may go on to find ϕ_1 by the method of variation of parameters if one knows γ_0 and ϕ_0, and hence ψ_0, explicitly.

It so happens that, if

$$N^2 = \operatorname{sech}^2 z \quad \text{for} \quad -\infty < z < \infty, \tag{26}$$

it can easily be verified that

$$\gamma_0 = 2/\{(2n-1+2\alpha)^2 - 1\}^{\frac{1}{2}}, \quad \phi_0 = \operatorname{sech}^{\alpha} z\, C_{n-1}^{(\alpha+\frac{1}{2})}(\tanh z) \quad \text{for} \quad n = 1, 2, \ldots, \tag{27}$$

where $C_{n-1}^{(\alpha+\frac{1}{2})}$ is the Gegenbauer or ultraspherical polynomial of degree $n-1$ (cf. Abramowitz & Stegun 1965, chap. 22). This solution was found by Groen (1948), who incidentally derived a perturbation to estimate the change in γ_0 due to the effects of the variation in density on the inertia of the fluid (which we have neglected from the start in the Taylor–Goldstein equation). This distribution (26) of the Brunt–Väisälä frequency corresponds to a mean density distribution of the form $\bar{\rho} = \rho_0 \exp(-b \tanh z)$, which can, by judicious choice of the parameters ρ_0 and b, describe fairly well many distributions met in practice. (One may also note that the above eigenfunctions ϕ_0 vanish at $z = 0$ if n is an even integer, and so are appropriate for semi-bounded flows over the domain $z > 0$.) The explicit solution (27) enables us to apply in detail the formula (25) to any unbounded flow with a well-behaved velocity profile.

4. Numerical results for particular flows

Taking the special basic flow with

$$U = \sin z, \quad N^2 = 1 \quad \text{for} \quad -\pi \leqslant z \leqslant \pi \tag{28}$$

as an example, we apply the previous results. In this way properties of a general flow will be illustrated. Here there is just one unstable mode, it being unstable for the parametric ranges

$$0 < \alpha^2 < \tfrac{3}{4}, \quad 0 \leqslant J < (1-\alpha^2)^{\frac{1}{2}} - 1 + \alpha^2 \equiv J_1(\alpha) \tag{29}$$

(cf. Huppert 1973, §2.1). This is shown in figure 14 of Hazel (1972). The eigensolutions on the marginal curves are given by

$$c = 0, \quad \phi = (\sin z)^{(1-\alpha^2)^{\frac{1}{2}}} \quad \text{on} \quad J = J_1(\alpha), \tag{30}$$

and
$$c = 0, \quad \phi = (\cos \tfrac{1}{2}z)^{\frac{1}{2}\pm\nu}(\sin \tfrac{1}{2}z)^{\frac{1}{2}\mp\nu} \quad \text{on} \quad \alpha^2 = \tfrac{3}{4}, \tag{31}$$

where
$$\nu = +(\tfrac{1}{4} - J)^{\frac{1}{2}}.$$

To find the discrete class of neutrally stable modes, namely the internal gravity waves modified by the basic shear, we first apply formulae (5), (11), (18) and (22) and find that $\gamma_{2m-1} = 0$ for all positive integers m and that

$$c = c_n = \pm J^{\frac{1}{2}}(\alpha^2 + \tfrac{1}{4}n^2)^{-\frac{1}{2}}\{1 + \tfrac{1}{8}J^{-1}[(2+\delta_{n2})(3\gamma_0^{-2} - 1) + 2(1-2\gamma_0^{-2})^2/(1-n^2+\delta_{n1})] + O(J^{-2})\} \tag{32}$$

as $J \to \infty$ for fixed α^2 and n.

We also solved the problem by direct numerical integration of the Taylor–Goldstein equation, using the shooting method to compute the eigenvalues and eigenfunctions for moderately wide ranges of α, J and n. The method adopted here was to use the results calculated from the asymptotic result (32) as trial values for large J. It was then possible to proceed to smaller values of J by using the results predicted by (32) in conjunction with the numerical solutions obtained for larger values of J.

Some typical characteristics of the modified internal gravity waves are shown in figures 1–3. It can be seen that formula (32) is quite accurate for $n = 1$ and $\alpha^2 = \frac{3}{4}$, even when J is as small as 3. It can also be seen that $c \to 1$ and a singularity at $z = \frac{1}{2}\pi$ develops as $J \to 0$;† this property, found for all modes examined by us, is analysed in the next section. Only the positive values of c have been shown, because there is symmetry between modes with positive and negative values of c. This symmetry of the flow (28) means that for each wave travelling in one direction there is a similar wave with equal but opposite velocity. It can be inferred by noting that the existence of an eigensolution $(c, \phi(z))$ implies the existence of another $(-c, \phi(-z))$. The integer n used to count the discrete eigensolutions (5) serves to count the eigensolutions for $J < \infty$, giving the number of antinodes of the eigenfunction between the walls. Thus the nth eigenfunction has $n-1$ zeros between the walls (at $z = \pm\pi$).

On the basis of the numerical results corresponding to $n = 1, 2$ and 3 for various values of J it was conjectured that

$$c = 1 + 2J/n(n+2) + o(J) \tag{33}$$

as $J \to 0$. The $n = 4$ mode was then examined numerically near $J = 0$, and the results were consistent with this conjecture.

As a second example we consider the stability of the jet given by

$$U = \operatorname{sech}^2 z, \quad N^2 = \operatorname{sech}^2 z \quad \text{for} \quad -\infty < z < \infty. \tag{34}$$

The marginal-stability curve for the sinuous mode of this jet is given by

$$J = \tfrac{1}{9}\alpha^2(2-\alpha)(3-\alpha) \quad \text{for} \quad 0 \leqslant \alpha \leqslant 2, \tag{35}$$

on which
$$c = \tfrac{1}{3}\alpha, \quad \phi = (\operatorname{sech} z)^{\alpha}(\operatorname{sech}^2 z - \tfrac{1}{3}\alpha)^{(1-\frac{1}{2}\alpha)}.$$

† No numerical difficulties were encountered, although, for fixed α^2, as $J \to 0$ the integrations took longer to perform.

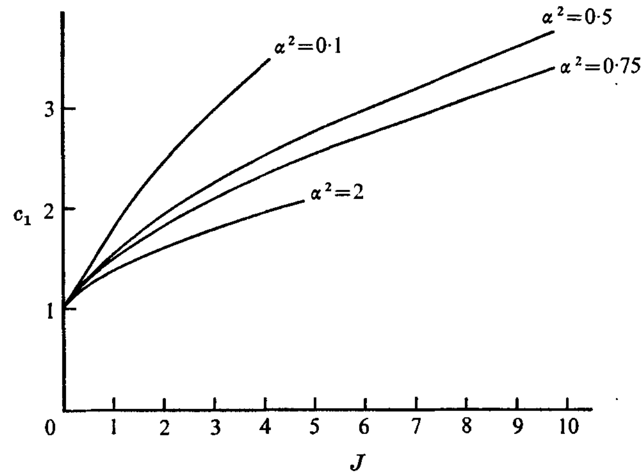

FIGURE 1. Sinusoidal flow (28): c *vs*. J for $n = 1$ and various values of α^2. ——, numerical results.

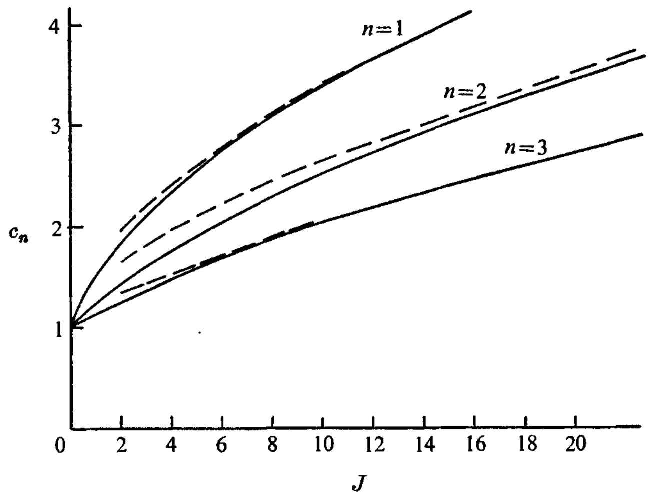

FIGURE 2. Sinusoidal flow (28): c *vs*. J for $n = 1, 2, 3$ and $\alpha^2 = \frac{3}{4}$. ——, numerical results; – – – –, large-J expansion.

The curve for the varicose mode is given by

$$J = \alpha(1-\alpha)(3-\alpha)(3+\alpha^2)/9(1+\alpha) \quad \text{for} \quad 0 \leqslant \alpha \leqslant 1, \tag{36}$$

on which

$$c = (3+\alpha^2)/3(1+\alpha), \quad \phi = \tanh z(\operatorname{sech} z)^\alpha (\operatorname{sech}^2 z - c)^{\frac{1}{2}(1-\alpha)}.$$

Proceeding as before, we find after using (25) to evaluate γ_1 in terms of beta functions that the velocities of the stable modes are given by

$$c = c_n = \pm\gamma_0 J^{\frac{1}{2}} + \{1-\gamma_0^2 + (3\gamma_0^2 - 1)/(2\alpha+3)\} + O(J^{-\frac{1}{2}}) \tag{37}$$

as $J \to \infty$ for fixed α^2 and n, where γ_0 is given in (27). We note that, for α, n and J fixed, there are two non-trivial solutions, corresponding to the plus or minus sign.

The Taylor–Goldstein equation was also integrated directly to compute the eigenvalues and eigenfunctions. Choosing $\alpha = \frac{1}{2}$, we obtained results for a

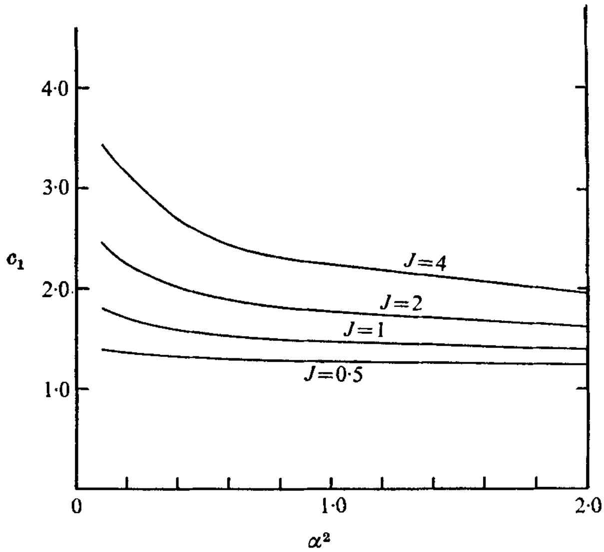

FIGURE 3. Sinusoidal flow (28): c *vs.* α^2 for $n = 1$ and various values of J.

number of values of J and n.† The procedure followed was that described earlier in this section, although behaviour according to the generalization (43) of (33) was anticipated in order to save computing time.

Some of the numerical results obtained for $\alpha = \frac{1}{2}$ are shown in figure 4. It may be noted that all the positive eigenvalues [corresponding to the plus sign in (37)] for the first three modes are greater than unity for $J > 0$, and that $c \downarrow 1$ as $J \downarrow 0$ with a resultant singularity in the limiting form of the eigenfunction (this corresponding to the simple maximum of $U = \operatorname{sech}^2 z$ at $z = 0$). The negative eigenvalues are also shown for the first three modes and we note that here $c \uparrow 0$ as $J \downarrow 0$; the singularity in this case arises at the minimum of U at $z = \pm\infty$, although the minimum is not a simple one [so formula (43) is not applicable]. The values predicted by just the first two terms of the large-J expansion are shown in figure 4 for comparison. With the exception of the third negative eigenvalue, the agreement is again good, even for moderately small values of J. The first two terms were further checked by calculating the positive and negative eigenvalues, say $c_\pm$, for one large value of J and comparing $c_+ - c_-$ with $2\gamma_0 J^{\frac{1}{2}}$ and $c_+ + c_-$ with $2\gamma_1$.

5. The internal gravity waves for small J

We found numerically that $c \to 1$ as $J \to 0$ for the flow (28) (and inferred a similar solution for which $c \to -1$ as $J \to 0$). To examine this analytically, first formally put $J = 0$ and $c = 1$ into the Taylor–Goldstein equation for the flow (28). Then one readily finds the solution with a singularity at $z = \frac{1}{2}\pi$, namely

$$\phi = A_1 f_1 + A_2 f_2,$$

† Certain results were also obtained with other values of α, but since these are all consistent with the theory set out here, they are not reproduced.

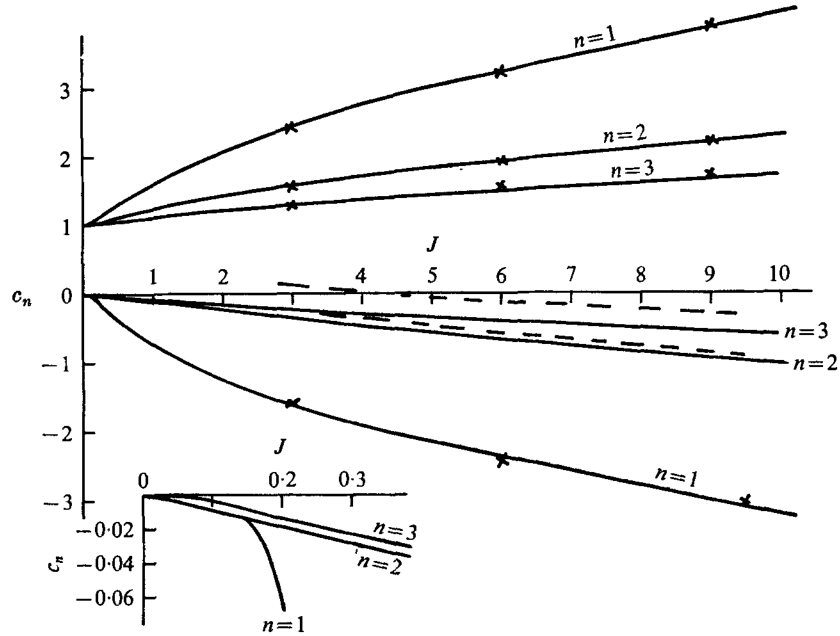

FIGURE 4. Jet flow (34): c *vs.* J for $n = 1, 2, 3$ and $\alpha = \frac{1}{2}$. ——, numerical results; – – – –, large-J expansion. (Where the results of the large-J expansion are close to the numerical results they are marked with a cross.) The inset is an enlargement of the lower quadrant near the origin.

where
$$f_1(z) \equiv (1-U)^{-\frac{1}{2}}g, \quad f_2 \equiv (1-U)^{-\frac{1}{2}}g\int(1-U)/g^2\,dz,$$
$$g(z) \equiv (a+\tfrac{1}{2})\cos\{(a-\tfrac{1}{2})(z+\tfrac{1}{2}\pi)\}+(a-\tfrac{1}{2})\cos\{(a+\tfrac{1}{2})(z+\tfrac{1}{2}\pi)\},$$
$$a \equiv +(1-\alpha^2)^{\frac{1}{2}}.$$

At this stage it seems best to confine our attention to the special case $\alpha^2 = \frac{3}{4}$ to illustrate more general behaviour, because then the above solution becomes so simple that we can explicitly satisfy the boundary conditions (2) to find

$$\phi = \begin{cases} A(1-U)^{-\frac{1}{2}}\{1+(z+\cos z)/(\pi+1)\} & \text{for} \quad -\pi \leqslant z < \frac{1}{2}\pi, \\ B(1-U)^{-\frac{1}{2}}\{1-(z+\cos z)/(\pi-1)\} & \text{for} \quad \frac{1}{2}\pi < z \leqslant \pi. \end{cases} \tag{38}$$

Note that we have not related B to A because of the singularity at $z = \frac{1}{2}\pi$, where $U = 1$. This singularity of the approximate solution (38) means that we face a singular perturbation problem and cannot simply expand the solution in powers of J.

We may proceed to obtain a uniformly valid expression for ϕ as $J \downarrow 0$ and $c \downarrow 1$ as follows. For fixed values $J > 0$ and $c > 1$ the Taylor–Goldstein equation (1) for the flow (28) has *two* complex conjugate critical layers at $z = z_c$, given by

$$z_c = \tfrac{1}{2}\pi \pm i\cosh^{-1}c. \tag{39}$$

These are the regular singularities of the equation. The theory of ordinary differential equations can readily be used to show that at each complex singularity

$$\phi \sim (z-z_c)^{\frac{1}{2}}\{D(z-z_c)^{\frac{1}{2}(1+4J/(c^2-1))^{\frac{1}{2}}}+E(z-z_c)^{-\frac{1}{2}(1+4J/(c^2-1))^{\frac{1}{2}}}\} \tag{40}$$

as $z \to z_c$ for fixed J and c, independently of the value of α^2, provided that the exponents do not differ by an integer.

Matching solutions (38) and (40) in the limit as $J \to 0$, and remembering that we have a real problem on the real axis, suggests that

$$\phi \sim E(c-U)^{-\frac{1}{2}} \quad \text{as} \quad z \to \tfrac{1}{2}\pi$$

and
$$\tfrac{1}{2} - \tfrac{1}{2}\{1+4J/(c^2-1)\}^{\frac{1}{2}} \to -\tfrac{1}{2} \quad \text{as} \quad J \to 0.$$

This in turn gives
$$c = 1 + \tfrac{2}{3}J + o(J) \tag{41}$$

and the uniformly valid eigenfunction

$$\phi \to \begin{cases} A(c-U)^{-\frac{1}{2}}\{1+(z+\cos z)/(\pi+1)\} & \text{for} \quad -\pi \leqslant z \leqslant \tfrac{1}{2}\pi \\ B(c-U)^{-\frac{1}{2}}\{1-(z+\cos z)/(\pi-1)\} & \text{for} \quad \tfrac{1}{2}\pi \leqslant z \leqslant \pi \end{cases} \tag{42}$$

as $J \to 0$. This argument is heuristic because (41) shows that the exponents in (40) differ by two in the limit as $J \to 0$.

We note that the result (41) is consistent with the conjecture (33), and also that the solution (42) is confirmed by our numerical calculations for small J only for the case $n = 1$. However, for all larger values of n it appears that we may regard (42) as an approximation as $J \to 0$ which breaks down near $z = \frac{1}{2}\pi$. We proceed to resolve the failure of (42) by regarding it as an 'outer' solution and investigating the 'inner' region in the immediate neighbourhood of $z = \frac{1}{2}\pi$.

We therefore suppose there is an eigenvalue such that $c = 1+\epsilon$, where $\epsilon \downarrow 0$ as $J \downarrow 0$, and seek the behaviour of the possible solutions of the Taylor–Goldstein equation near $z = \frac{1}{2}\pi$ by introducing an 'inner' variable ζ such that $z = \frac{1}{2}\pi + \delta\zeta$, where $\delta \to 0$ as $J \to 0$. The form of δ is obtained by requiring a balance of appropriate terms when substitution is made into (1) (with $N^2 = 1$ and $U = \sin z$); we find that $\delta = O(\epsilon^{\frac{1}{2}})$ and $\epsilon = O(J)$. We choose $\delta = (2\epsilon)^{\frac{1}{2}}$ for convenience.

Anticipating the numerical results, we define a constant n such that

$$\epsilon \sim 2J/n(n+2)$$

as $J \to 0$, and so the equation governing the structure near $z = \frac{1}{2}\pi$ is

$$\phi_{\zeta\zeta} + \left\{\frac{n(n+2)}{(1+\zeta^2)^2} - \frac{2}{1+\zeta^2}\right\}\phi = 0.$$

Note that this equation does not contain α.

It is convenient to write $x = i\zeta$ and $\psi = (1+\zeta^2)^{\frac{1}{2}n}\phi$ so that the equation for ψ becomes
$$(1-x^2)\,\psi_{xx} + 2nx\psi_x - (n-1)\,(n+2)\,\psi = 0.$$

From this equation we see that there are solutions which behave like x^{n-1} and x^{n+2} as $x \to \pm\infty$. We shall see later that we require $\phi = O(x^{-1})$ as $x \to \pm\infty$ and so must reject the solution ψ behaving like x^{n+2} as $x \to \pm\infty$. This gives $\psi \propto C_{n-1}^{(-n-\frac{1}{2})}(x)$, the ultraspherical polynomial (Abramowitz & Stegun 1965, chap. 22), and thence $\phi \propto (1+\zeta^2)^{-\frac{1}{2}n} C_{n-1}^{(-n-\frac{1}{2})}(i\zeta)$.

We can also use the equation for ψ in terms of ζ to calculate ψ directly for a few values of n. We find that

$$\phi_1 \propto (1+\zeta^2)^{-\frac{1}{2}}, \quad \phi_2 \propto (1+\zeta^2)^{-1}\zeta,$$
$$\phi_3 \propto (1+\zeta^2)^{-\frac{3}{2}}(\zeta^2 - \tfrac{1}{5}), \quad \phi_4 \propto (1+\zeta^2)^{-2}(\zeta^3 - \tfrac{3}{5}\zeta).$$

The function ϕ_n appears to have $n-1$ zeros in the interval $-1 < \zeta < 1$.

We conjecture that n must be a positive integer in order that ϕ behaves like ζ^{-1} at both $\zeta = -\infty$ and $\zeta = +\infty$; the matching of ϕ will be seen to give this behaviour, so that the eigenvalues of n are the positive integers on the basis of our conjecture.

It remains to match this inner solution to the outer solution given in (42). From the latter we find

$$\phi \sim \begin{cases} \dfrac{\frac{3}{2}\pi+1}{\pi+1} A\epsilon^{-\frac{1}{2}}(1+\zeta^2)^{-\frac{1}{2}} & \text{as} \quad z\uparrow \tfrac{1}{2}\pi, \quad J\downarrow 0 \\ \dfrac{\frac{1}{2}\pi-1}{\pi-1} B\epsilon^{-\frac{1}{2}}(1+\zeta^2)^{-\frac{1}{2}} & \text{as} \quad z\downarrow \tfrac{1}{2}\pi, \quad J\downarrow 0 \end{cases}$$

$$\sim \begin{cases} -\dfrac{\frac{3}{2}\pi+1}{\pi+1} A\epsilon^{-\frac{1}{2}}/\zeta & \text{as} \quad \zeta\to -\infty, \\ \dfrac{\frac{1}{2}\pi-1}{\pi-1} B\epsilon^{-\frac{1}{2}}/\zeta & \text{as} \quad \zeta \to +\infty. \end{cases}$$

But the inner solution gives

$$\phi_n \sim \begin{cases} (-1)^n a_n/\zeta & \text{as} \quad \zeta\to -\infty \\ a_n/\zeta & \text{as} \quad \zeta\to +\infty \end{cases}$$

for some constant a_n. The matching of the two solutions is achieved by requiring that

$$\left(\frac{\frac{1}{2}\pi-1}{\pi-1}\right) B\epsilon^{-\frac{1}{2}} = a_n, \quad (-1)^n a_n = -\left(\frac{\frac{3}{2}\pi+1}{\pi+1}\right) A\epsilon^{-\frac{1}{2}}$$

and so

$$B = (-1)^{n+1}\left(\frac{\pi-1}{\pi+1}\right)\left(\frac{3\pi+2}{\pi-2}\right) A.$$

This relationship between A and B agrees with the results deduced from the numerical integrations.

The inner solution derived above has been compared in detail with the numerical results and very good agreement obtained. This comparison included the positions of the zeros and the amplitudes of ϕ_n in the neighbourhood of $z = \frac{1}{2}\pi$ for $n = 1, 2, 3$ and 4 as $J \to 0$. This leaves no doubt that the analysis describes the behaviour completely.

The matching argument above can be seen to be independent of α and to depend upon the functions $U(z)$ and $N^2(z)$ only in the 'boundary layer'. Therefore the asymptotic eigenvalue relation can in fact be found similarly for any flow with an overall simple maximum or minimum of $U(z)$, say U_m at $z = z_m$. It is thus easy to show that

$$c = U_m - 2JN_m^2/\{n(n+2)\,U_m''\} + o(J) \quad \text{as} \quad J\to 0 \quad \text{for} \quad n = 1, 2, \ldots, \tag{43}$$

where $N_m = N(z_m)$ and $U_m'' = U''(z_m)$. However, the eigenfunction, and higher terms in the expansion of c for small J, depend upon α and more details of the structure of the functions $U(z)$ and $N(z)$. This generalization (43) as $J \to 0$ was verified numerically by considering (i) the $n = 2$ and $n = 3$ modes with $U = \sin z$ for $\alpha^2 = 0{\cdot}1$ and $2{\cdot}0$, (ii) the effect of changing from $N^2 = 1$ to $N^2 = \sin^2 z$, and (iii) the $n = 1, 2$, and 3 modes for $U = (z/\pi)(1-z^2/\pi^2)$, $N^2 = 1$ and $\alpha^2 = 0{\cdot}1$.

Our numerical results for the jet (34) verify further the universality of the formula (43). They also suggest that $(c-U_m)/J \to$ constant as $J \downarrow 0$ for fixed α and n, where $U_m = 0$, although this minimum of the velocity profile at infinity is not a simple one.

To investigate this we again use the method of matched asymptotic expansions. For the particular jet (34) it is convenient to define $T = \tanh z$. Then

$$U = N^2 = 1-T^2$$

and the Taylor–Goldstein problem becomes

$$(1-c-T^2)\left\{(1-T^2)\frac{d}{dT}\left[(1-T^2)\frac{d\phi}{dT}\right]-\alpha^2\phi\right\} + 2(1-3T^2)(1-T^2)\phi + \frac{J(1-T^2)}{1-c-T^2}\phi = 0, \quad (44)$$

$$\phi = 0 \quad \text{at} \quad T = \pm 1. \quad (45)$$

For the outer solution over the region $-1 < T < 1$ we put $c = J = 0$ formally, and derive the outer equation

$$(1-T^2)\frac{d}{dT}\left\{(1-T^2)\frac{d\phi_0}{dT}\right\}-\alpha^2\phi_0+2(1-3T^2)\phi_0 = 0. \quad (46)$$

This has general solution in terms of hypergeometric functions

$$\phi_0 = A(1-T^2)^r F(3+2r, 2r-2;\ 1+2r;\ \tfrac{1}{2}-\tfrac{1}{2}T) + B(1-T^2)^{-r} F(3-2r, -2r-2;\ 1-2r;\ \tfrac{1}{2}-\tfrac{1}{2}T),$$

where $r = +(1+\frac{1}{4}\alpha^2)^{\frac{1}{2}}$ and A and B are arbitrary constants.

It is convenient to reduce the domain of the problem to the half-interval $0 \leqslant T \leqslant 1$ by imposing a symmetry condition at $T = 0$. This is possible for any problem for which U and N^2 are even functions. Thus for a sinuous or even mode (n odd) we require

$$d\phi/dT = 0 \quad \text{at} \quad T = 0.$$

This can be shown at length to give

$$A = 2^{-4r}\frac{(r+1)(2r+1)}{(r-1)(2r-1)}B, \quad (47)$$

after use of several formulae for the hypergeometric and gamma functions (see, for example, Abramowitz & Stegun 1965, chaps. 6, 15). Similarly, for a varicose or odd mode (n even), we require

$$\phi = 0 \quad \text{at} \quad T = 0,$$

and find

$$A = -2^{-4r}\frac{(r+1)(2r+1)}{(r-1)(2r-1)}B. \quad (48)$$

An inner solution is needed because the outer solution is singular near $T = 1$. Examination of the balance of terms in the Taylor–Goldstein equation (44) for small J and $1-T$ but for fixed $\alpha > 0$ and J/c suggests that a suitably stretched inner variable may be defined by $t = 2(1-T)/c$. Then the inner equation can be seen to be

$$\frac{d^2\phi_i}{dt^2}+\frac{1}{t}\frac{d\phi_i}{dt}+\left\{-\frac{\alpha^2}{4t^2}-\frac{1}{t(t-1)}+\frac{\frac{1}{4}J/c}{t(t-1)^2}\right\}\phi_i = 0. \quad (49)$$

Any solution of this equation which vanishes at $t = 0$ (i.e. $T = 1$ or $z = \infty$) has the form

$$\phi_i = D(-t)^{\frac{1}{2}\alpha}(1-t)^{\frac{1}{2}+\frac{1}{2}s}F(\tfrac{1}{2}\alpha+\tfrac{1}{2}+\tfrac{1}{2}s+r, \tfrac{1}{2}\alpha+\tfrac{1}{2}+\tfrac{1}{2}s-r;\ 1+\alpha;\ t)$$

for some constant D of normalization, where $s = +(1-J/c)^{\frac{1}{2}}$. In our numerical work we chose to normalize such that $\phi \sim (1-T)^{\frac{1}{2}\alpha}$ as $T \to 1$, and this gives

$$D = (-\tfrac{1}{2}c)^{\frac{1}{2}\alpha}. \tag{50}$$

The matching condition is that

$$\lim_{T\to 1}\phi_0 = \lim_{t\to -\infty}\phi_i.$$

This gives, after use of a connexion formula for the hypergeometric functions of ϕ_i and of the poles of the gamma function, both the eigenvalue relation

$$s = 2j+1+\alpha+2r \quad \text{for} \quad j = 0, 1, 2, \ldots,$$

and the eigenfunction by means of the formula

$$B = (-c)^r \frac{\Gamma(1+\alpha)\,\Gamma(-2r)}{\Gamma(1+\alpha+j)\,\Gamma(-j-2r)}D. \tag{51}$$

It follows that

$$c = -\frac{J}{(2j+1+\alpha)^2+3+\alpha^2+4(2j+1+\alpha)\,r}+o(J) \quad \text{as} \quad J\downarrow 0 \quad \text{for fixed} \quad \alpha > 0, \tag{52}$$

where $j = \frac{1}{2}(n-1)$ for a sinuous mode and $j = \frac{1}{2}(n-2)$ for the next higher varicose mode. Thus each value of j gives the same value of $\lim_{J\to 0} c/J$ for the pair of modes. The eigenfunctions of the pair of modes do differ, however, with

$$\phi_0 = 2^{-2r-\frac{1}{2}\alpha}(-c)^{\frac{1}{2}\alpha+r}\frac{(2r+1)\,\Gamma(1+\alpha)\,\Gamma(-2r)}{(r-1)\,\Gamma(1+\alpha+j)\,\Gamma(-j-2r)}, \quad \frac{d\phi_0}{dT} = 0 \quad \text{at} \quad T = 0 \tag{53}$$

for the sinuous modes and

$$\phi_0 = 0, \quad \frac{d\phi_0}{dT} = 2^{3-2r-\frac{1}{2}\alpha}(-c)^{\frac{1}{2}\alpha+r}\frac{r(r+1)\,\Gamma(1+\alpha)\,\Gamma(-2r)}{(2r-1)\,\Gamma(1+\alpha+j)\,\Gamma(-j-2r)} \quad \text{at} \quad T = 0 \tag{54}$$

for the varicose modes.

The inner solution here has again been checked against the numerical results and very good agreement found. The comparison with the numerical results included not only a direct check on the eigenvalue relation (52) but also a comparison of the values of ϕ_0 and $d\phi_0/dT$ as given in (53) and (54) for the sinuous and varicose modes respectively.

6. The applicability of Howard's perturbation formula

The example (28) of a sinusoidal flow offers an opportunity to consider a general aspect of the theory of unstable modes. As for many examples, the stability boundary for the flow (28) has been found analytically. Further, Howard (1963) derived the general formula

$$(\partial c/\partial \alpha^2)_J = \lim_{c_i \downarrow 0}(I_0/I_1) \tag{55}$$

to perturb the marginal-stability characteristics, where

$$\left.\begin{aligned} I_0 &= \int_{z_1}^{z_2} \phi^2 dz, \quad I_1 = I_{11} + 2JI_{12}, \\ I_{11} &= -\int_{z_1}^{z_2} U''\phi^2/(U-c)^2 dz, \quad I_{12} = \int_{z_1}^{z_2} N^2\phi^2/(U-c)^3 dz. \end{aligned}\right\} \tag{56}$$

So much has been related by Huppert (1973), who went on to question the validity of Howard's formula in various cases. For one of these cases, Banks & Drazin (1973, § 5) resolved the singularity where Howard's formula gives $\partial c/\partial \alpha^2 = \infty$ because in fact $c \sim \text{constant} \times (\alpha_0^2 - \alpha^2)^{\frac{1}{2}}$ as $\alpha^2 \uparrow \alpha_0^2$ for a fixed value of J such that $c(\alpha_0, J) = 0$. Here we show that the detailed structure of the eigensolution may be required if one or both of the integrals I_0 and I_1 is zero or infinite, or if there is a critical layer near a boundary. The singular nature of the perturbation must be recognized.

Thorpe's eigensolutions for flow (28) are given by (31). Huppert (1973) applied the Howard formula (55) to derive

$$\partial c/\partial \alpha^2 = \mp 2\nu i \quad \text{on} \quad \alpha^2 = \tfrac{3}{4} \tag{57}$$

essentially, and pointed out that the lower sign gave results in conflict with numerical evidence.

We are unable to recover Huppert's results (57), finding instead that I_{12} is infinite in the limit as $c_i \downarrow 0$, whereas

$$I_0 = \mp 2\pi i\nu e^{\pm\pi\nu i}, \quad I_{11} = \pi e^{\pm \pi\nu i}.$$

The difficulty in the evaluation of I_{12} is that there are three critical layers, two of which are near the walls. Therefore, to estimate the integral I_{12} asymptotically as $c_i \downarrow 0$ we used the method of matching to derive the following solution as $\alpha^2 \uparrow \frac{3}{4}$ $(J \neq \frac{1}{4})$:

$$\phi \sim \left\{\begin{array}{lll} \phi_- = (c/2\nu\pi)(\zeta_-^{\frac{1}{2}+\nu} - \zeta_-^{\frac{1}{2}-\nu}) & \text{near} & z = -\pi, \\ \Phi_- = \dfrac{\exp(\frac{1}{4}\pi i - \frac{1}{2}\pi\nu i)(\frac{1}{2}c_i)^{\frac{1}{2}-\nu}}{\nu\pi}(-\sin\frac{1}{2}z)^{\frac{1}{2}-\nu}(\cos\frac{1}{2}z)^{\frac{1}{2}+\nu} & \text{for} & -\pi < z < 0, \\ \phi_0 = (\frac{1}{2}c_i)^{1-2\nu}\zeta_0^{\frac{1}{2}-\nu}/\nu\pi & \text{near} & z = 0, \\ \Phi_+ = \dfrac{\exp(\frac{3}{4}\pi i - \frac{3}{2}\pi\nu i)(\frac{1}{2}c_i)^{\frac{1}{2}-\nu}}{\nu\pi}(\sin\frac{1}{2}z)^{\frac{1}{2}-\nu}(\cos\frac{1}{2}z)^{\frac{1}{2}+\nu} & \text{for} & 0 < z < \pi, \\ \phi_+ = (ce^{-2\pi\nu i}/2\nu\pi)(\zeta_+^{\frac{1}{2}+\nu} - \zeta_+^{\frac{1}{2}-\nu}) & \text{near} & z = \pi, \end{array}\right\} \tag{58}$$

where the inner variables are defined such that

$$\zeta_\mp = (z \pm \pi + c)/c, \quad \zeta_0 = (z-c)/c$$

asymptotically. Note that the *upper* signs in the solutions (31) have been taken to give the outer solutions $\Phi_\mp$ because it is not possible to match the solutions with the lower signs; also we have renormalized the solutions (31) such that $d\phi/dz = \pi^{-1}$ at $z = -\pi$. It seems essential to use the *two* terms in each of the inner solutions $\phi_\mp$ for the critical layers near the walls.

It can now be shown that the dominant contribution to the integral I_{12} as $c_i \downarrow 0$ comes from the two inner solutions near the walls and that

$$I_{12} \to -i\,e^{-2\nu\pi i} \sin(2\nu\pi)/\pi^2 J \quad \text{as} \quad c_i \downarrow 0. \tag{59}$$

For fixed $J \neq 0$ or $\frac{1}{4}$, I_{12} dominates I_{11} in the limit as $c_i \downarrow 0$, although it transpires that as J decreases the contribution from I_{11} becomes relatively more important for fixed α^2 near $\frac{3}{4}$ and the term I_{11} must be retained. It follows from formula (55) that

$$\frac{\partial c_i}{\partial \alpha^2} \to \frac{-2\pi\nu}{\pi + 2\nu^2 \sin 2\nu\pi (\frac{1}{2}c_i)^{-1+2\nu}} \quad \text{as} \quad c_i \downarrow 0,$$

and therefore the equation determining c_i is

$$c_i + 2\nu\pi^{-1} \sin 2\nu\pi (\tfrac{1}{2}c_i)^{2\nu} = 2\nu(\tfrac{3}{4} - \alpha^2) \quad \text{as} \quad \alpha^2 \uparrow \tfrac{3}{4}. \tag{60}$$

Equation (60) gives

$$c_i \sim 2\{\pi(\tfrac{3}{4} - \alpha^2)/\sin 2\pi\nu\}^{1/2\nu} \quad \text{as} \quad \alpha^2 \uparrow \tfrac{3}{4} \quad \text{if} \quad \nu \neq 0 \text{ or } \tfrac{1}{2},$$

although in some of the cases where numerical results were obtained the contribution from the first term on the left-hand side of (60) was significant. The logical justification for the result (60) is poor, because Howard's formula (55) is derived on the basis of a regular perturbation. However, our numerical results, found by direct integration of the eigenvalue problem, support (60) as will be seen from table 1, where c_i is tabulated for $\alpha^2 = 0{\cdot}749$ at various values of J. When J is close to $\frac{1}{4}$ a transcendental singularity seems to develop; in fact our numerical results lead us to conjecture tentatively that

$$c_i \sim A \exp\{-B(\tfrac{3}{4} - \alpha^2)^{-\frac{1}{2}} (\tfrac{1}{4} - J)^{-\frac{1}{2}}\} \quad \text{as} \quad \alpha^2 \uparrow \tfrac{3}{4}, \quad J \uparrow \tfrac{1}{4}, \tag{61}$$

for some constants A and B.

We may note here that certain of the eigenvalues were calculated using the Howard equation [Howard 1963, equation (10)] for $H = (U-c)^{n-1}\phi$, where n is evaluated from the relation $n(1-n)\,U_c'^2 = JN^2$ and the subscript c indicates evaluation at the critical layer near $z = 0$. For eigenvalues calculated near the singular neutral modes this method reduced computation time considerably since the most important singular-like behaviour in ϕ is removed. Confirmation of the behaviour of I_{12} as given by (59) was obtained independently by evaluating I_{12} numerically for $J = 0{\cdot}23$, $0{\cdot}1875$ and $0{\cdot}1$ with $\alpha^2 = 0{\cdot}749$; very good agreement was found for each value.

On the other branch of the marginal curve (29), where the eigensolution is given by (30) for $0 < \alpha^2 < \frac{3}{4}$, our numerical results are consistent† with the Howard perturbation formulae as evaluated by Huppert [1973, equation (2.4)].

† The weak phrasing here arises because of the behaviour of c near $J = J_1$: the numerical results are highly suggestive of a singular second derivative of c with respect to J (or α^2 of course), so that the approach of $\partial c/\partial J$ to the Howard value as $J \uparrow J_1$ is of cusp form. This could arise because of the occurrence of a term like $(J_1 - J)^p$ in $\partial c/\partial J$ near $J = J_1$, where $0 < p < 1$, although other possibilities clearly exist and no special computations were performed to seek the nature of this singularity.

J		0·02	0·10	0·16	0·20	0·23
c_i	numerical	$9{\cdot}32\times10^{-4}$	$5{\cdot}09\times10^{-4}$	$1{\cdot}05\times10^{-4}$	$5{\cdot}04\times10^{-6}$	$6{\cdot}76\times10^{-9}$
	equation (60)	$9{\cdot}34\times10^{-4}$	$5{\cdot}11\times10^{-4}$	$1{\cdot}06\times10^{-4}$	$5{\cdot}20\times10^{-6}$	$6{\cdot}90\times10^{-9}$

TABLE 1. Comparison of numerical and analytical results for c_i for $\alpha^2 = 0{\cdot}749$

7. Discussion

It should be noted that the result (43) for small values of the Richardson number is widely applicable, but only to flows with a simple maximum or minimum. This suggests that the spectrum of modified internal gravity waves for monotone velocity profiles may be qualitatively different. However, the results of Miles (1967) for a particular monotone semi-bounded flow with a particular density distribution disagree with the precise form of (43), although the results for small values of J are qualitatively similar.

The numerical results of § 5 which led to our finding modes for which $c \to 1$ as $J \to 0$ came as a surprise to us, because at first sight they seem at variance with the conclusions of § 2. The numerical and asymptotic analysis of § 5 suggests that there are two infinite complete sets of modified internal gravity waves as $J \to 0$, which are, plausibly, the continuation of the complete sets which were found in § 3 as $J \to \infty$. The asymptotic analysis offers no proof of this, however, because it is valid as $J \to 0$ only for fixed n.

This leads us to re-examine the basis for the conclusion of § 2 that there is only a finite number of modified internal gravity waves for each finite value of J. This conclusion was based upon the general spectral theory used by Dikiy & Katayev (1971) and upon what treatment of the initial-value problem there is in the literature. Further, Eliassen *et al.* (1953; cf. Drazin & Howard 1966) found the continuous spectrum for the specific case of plane Couette flow, unbounded, bounded and semi-bounded, there being no discrete modes when $0 < J < \frac{1}{4}U'^2$; however, this result is for a basic flow which has neither a simple maximum nor a simple minimum in the field of flow. Other authors have found various isolated stable modes, with c in the range of $U(z)$, for various basic flows (see, for example, (30) with $\alpha^2 > \frac{3}{4}$).

It would seem that the assumptions of these arguments need to be questioned. In particular, a more thorough examination of the initial-value problem is required in order to resolve the issue, because the normal modes for the singular equation may not be independent. Moreover, the initial-value problem is not very well understood in view of the meagre treatment it has been given. There is also a practical and theoretical need to relate the stability characteristics for inviscid non-diffusive fluid to those for a slightly viscous and slightly diffusive fluid.

The details of the example of § 6 were examined less for their own sake than to show with what caution Howard's formula should be used, just as § 5 was directed towards the generality of formula (43). Indeed, the instability discussed in § 6 shows that the internal gravity waves with velocities (33) are unlikely to be observed in practice because they would be overwhelmed by turbulence. However, for flows which are stable when $J > 0$, the internal gravity waves with velocities (43) would not be obscured by any instability and so would be observable, particularly when a source, such as a rigid body, moves at the maximum or minimum velocity of the mean flow.

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Addendum

After this paper had been accepted for publication another class of flows was investigated in the same spirit as indicated above. We considered those basic flows in which the overall maximum or minimum value of the velocity occurred at the boundary (with non-zero shear there). As a typical case we examined the characteristics of the flow defined by

$$U(z) = z^3 - z, \quad N^2 = 1, \quad -2 \leqslant z \leqslant 2, \quad \text{for} \quad \alpha = 1$$

by numerically integrating the Taylor–Goldstein equation as outlined above.

The large-J analysis generated by the modified form of (5) etc. of §§2 and 3 was again used. The first two modes were evaluated for large values of J and then with successively smaller values until evidence of a singularity was found. The results of these integrations are shown in figure 5, where we have displayed values of $U(-2) - c = -6 - c$ for various J (only the positive eigenvalues are shown). We again found that for large values of J the behaviour predicted by §3 is confirmed.†

For the first mode we found that the singularity occurs at $J \doteqdot 25{\cdot}6$. No special calculations were done to evaluate this critical value of J very accurately, although we are satisfied that it is correct to three significant figures, and that $U(-2) - c$ vanishes algebraically. We presume that for J less than this critical value all modes belong to the continuous spectrum. For the second mode, however, no critical value of J could be found numerically, at least not by the method we have used, because of the very rapid approach to zero of $U(-2) - c$: the numerical results strongly suggest an exponential decay as may be inferred from figure 5.

In the light of these remarks we may anticipate that, for a general distribution $U(z)$ in the interval (z_1, z_2) with $U_1 \equiv U(z_1) < U(z)$ and $U_1' \equiv U'(z_1) > 0$, $c_n \uparrow U_1$ as $J \downarrow J_n$ for $n = 1, 2, \ldots$, where the problem determining $J_n = J_n(\alpha)$ will be posed later. With this as the basic hypothesis we proceed to construct a uniformly valid approximation as $c_n \uparrow U_1$ and $J \downarrow J_n$.

† Since the chosen $U(z)$ is an odd function it follows that $\gamma_1 = \gamma_3 = \ldots = 0$, and although γ_0 is easily found we have not attempted to evaluate γ_2 analytically. However, if we assume $c = aJ^{\frac{1}{2}} + bJ^{-\frac{1}{2}}$ and evaluate a and b by using those values of c found numerically for two large values of J, we find that for both the first and the second mode the resulting values of a agree very closely with those given by γ_0 from the modified form of (5). For comparison we also plot $c = aJ^{\frac{1}{2}} + bJ^{-\frac{1}{2}}$ in figure 5 for both modes.

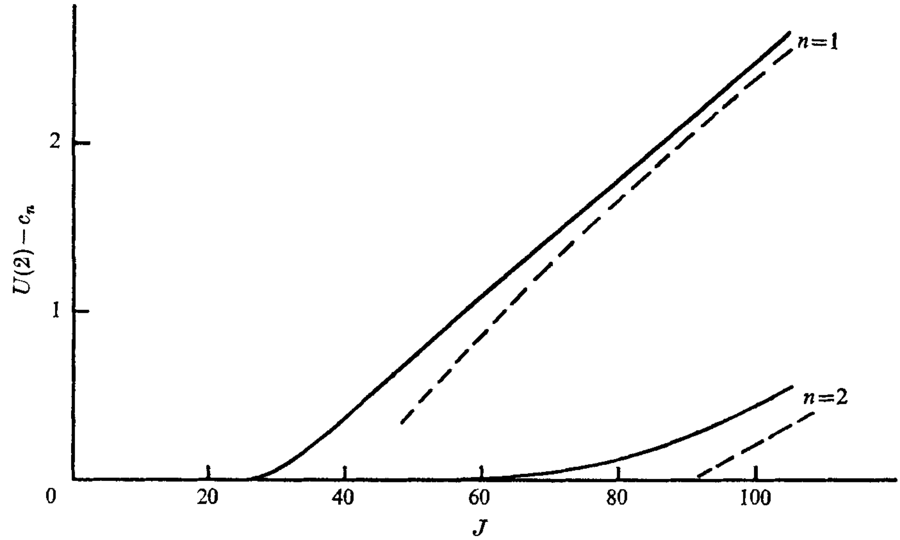

FIGURE 5. $U(z) = z^3-z$, $N^2 = 1$, $-2 \leqslant z \leqslant 2$: $U(-2)-c_n$ *vs.* J for the first two modes with $\alpha = 1$. ——, numerical results; – – –, $c = aJ^{\frac{1}{2}}+bJ^{-\frac{1}{2}}$ (see text).

Thus we formally put $c = U_1$ into the Taylor–Goldstein equation to derive the outer equation,

$$(U-U_1)(\phi_0''-\alpha^2\phi_0)-U''\phi_0+JN^2\phi_0/(U-U_1) = 0, \tag{62}$$

which is non-singular for $z_1 < z \leqslant z_2$. The boundary condition

$$\phi_0 = 0 \quad \text{at} \quad z = z_2 \tag{63}$$

specifies ϕ_0 uniquely except for an arbitrary constant of normalization.

Examination of the balance of terms of the Taylor–Goldstein equation near the singularity $z = z_1$ of the outer equation with $U-U_1 \sim U_1'(z-z_1)$ suggests that we choose the inner variable as $Z = U_1'(z-z_1)/(U_1-c)$. Thus in the limit as $c \uparrow U_1$ for fixed Z and J we derive the inner equation,

$$(Z+1)^2\frac{d^2\phi_i}{dZ^2}+\frac{JN_1^2}{U_1'^2}\phi_i = 0, \tag{64}$$

where $N_1 = N(z_i)$. The inner solution must vanish at $z = z_1$; it is also convenient to normalize such that $d\phi_i/dz = 1$ at $z = z_1$. These conditions give the unique inner solution

$$\phi_i = \frac{U_1-c}{2\nu U_1'}(Z+1)^{\frac{1}{2}}\{(Z+1)^{\nu}-(Z+1)^{-\nu}\}, \tag{65}$$

where we define $\nu = (\frac{1}{4}-JN_1^2/U_1'^2)^{\frac{1}{2}}$.

To match ϕ_0 and ϕ_i we first take the inner limit as $z \downarrow z_1$ of the outer solution. Although we cannot in general express ϕ_0 in terms of well-known functions, the theory of ordinary differential equations gives

$$\phi_0 \sim (z-z_1)^{\frac{1}{2}}\{A(z-z_1)^{\nu}+B(z-z_1)^{-\nu}\} \quad \text{as} \quad z \downarrow z_1$$

for some constants A and B whose ratio is determined uniquely by the outer problem (62) with (63). The matching depends crucially upon whether ν is real or pure imaginary. First consider the case $J < \frac{1}{4}U_1'^2/N_1^2$ and take $\nu > 0$ for definiteness. Then

$$\phi_i \sim \frac{(U_1-c)}{2\nu U_1'} Z^{\nu+\frac{1}{2}} = \frac{1}{2\nu}\left(\frac{U_1-c}{U_1'}\right)^{\frac{1}{2}-\nu}(z-z_1)^{\nu+\frac{1}{2}} \quad \text{as} \quad Z\uparrow\infty.$$

Therefore matching $\lim_{z\downarrow z_1}\phi_0 = \lim_{Z\uparrow\infty}\phi_i$, we require that

$$A = \frac{1}{2\nu}\left(\frac{U_1-c}{U_1'}\right)^{\frac{1}{2}-\nu} \quad \text{and} \quad B/A = 0 \quad \text{to this order.}$$

It follows that

$$\phi_0 \sim \frac{1}{2\nu}\left(\frac{U_1-c}{U_1'}\right)^{\frac{1}{2}-\nu}(z-z_1)^{\nu+\frac{1}{2}} \quad \text{as} \quad z\downarrow z_1. \tag{66}$$

In this case of real ν, we see that (62), (63) and (66) pose a singular Sturm–Liouville problem to determine the limiting eigenvalues J as $c\uparrow U_1$ for any fixed value of α. There is a finite number (possibly zero) of these values $J < \frac{1}{4}U_1'^2/N_1^2$, say $J_1, J_2, \ldots, J_p$. Further, if α is sufficiently large then the solution ϕ_0 is exponential in character, so that p is certainly zero.

To find how $c\uparrow U_1$ as $J\downarrow J_n$ we must go to a higher approximation. Now, on expanding A and B in powers of $J-J_n$, since we may expect both to be regular, the outer solution can be put in the form

$$\phi_0 = \frac{1}{2\nu_n}\left(\frac{U_1-c}{U_1'}\right)^{\frac{1}{2}-\nu_n}(z-z_1)^{\frac{1}{2}}\{[1+(J-J_n)A_1](z-z_1)^{\nu_n}$$
$$+(J-J_n)B_1(z-z_1)^{-\nu_n}\}+O(J-J_n)^2 \quad \text{as} \quad z\downarrow z_1, \quad J\downarrow J_n,$$

where $\nu_n = +(\frac{1}{4}-J_nN_1^2/U_1'^2)^{\frac{1}{2}}$. Then matching with the outer limit of the inner solution (65) gives

$$U_1-c \sim U_1'\{-B_1(J-J_n)\}^{1/2\nu_n} \quad \text{as} \quad J\downarrow J_n, \tag{67}$$

for some B_1 determined only by the outer problem.

We note here that the numerical results are consistent with the formula (67) with $J_1 \doteqdot 25{\cdot}6$ for the first mode only (i.e. $p=1$). The critical value $J_1 \doteqdot 25{\cdot}6$ was found by plotting the curves $\log(J-J_1)$ against $\log(U_1-c)$ for various values of J_1 and choosing J_1 by requiring the curve to be a straight line.†

Second, we consider the case of pure imaginary ν, i.e. $J > \frac{1}{4}U_1'^2/N_1^2$. It is convenient then to define $\mu = +(JN_1^2/U_1'^2-\frac{1}{4})^{\frac{1}{2}}$ and rewrite the inner solution (65) as

$$\phi_i = \frac{U_1-c}{\mu U_1'}(Z+1)^{\frac{1}{2}}\sin\{\mu\log(Z+1)\}. \tag{68}$$

† We may note that for $U = z^3-z$ with $\alpha = 1$ only one mode exists with ν real, although further numerical work suggests that for $\alpha < \alpha^*$ where $\alpha^* > \frac{3}{4}$ two modes exist with ν real.

Therefore

$$\lim_{Z\uparrow\infty}\phi_i \sim \frac{U_1-c}{\mu U_1'} Z^{\frac{1}{2}} \sin\{\mu \log Z\}$$

$$= \frac{1}{\mu}\left(\frac{U_1-c}{U_1'}\right)^{\frac{1}{2}}(z-z_1)^{\frac{1}{2}}\left[\sin\{\mu\log(z-z_1)\}\cos\left\{\mu\log\left(\frac{U_1-c}{U_1'}\right)\right\}\right.$$

$$\left. -\cos\{\mu\log(z-z_1)\}\sin\left\{\mu\log\left(\frac{U_1-c}{U_1'}\right)\right\}\right].$$

Examination of ϕ_0 as $z \downarrow z_1$ reveals that $\mu \downarrow 0$ in order that matching may be possible. Then we find

$$\lim_{z\downarrow z_1}\phi_0 \sim (z-z_1)^{\frac{1}{2}}\{A'+B'\log(z-z_1)\}.$$

where $A' = A+B$ and $B' = i\mu\,(A-B)$, and deduce that

$$A' = -\frac{1}{\mu}\left(\frac{U_1-c}{U_1'}\right)^{\frac{1}{2}}\sin\left\{\mu\log\left(\frac{U_1-c}{U_1'}\right)\right\},$$

$$B' = \left(\frac{U_1-c}{U_1'}\right)^{\frac{1}{2}}\cos\left\{\mu\log\left(\frac{U_1-c}{U_1'}\right)\right\}.$$

Therefore

$$\tan\left\{\mu\log\left(\frac{U_1-c}{U_1'}\right)\right\} \sim -\frac{\mu A'}{B'} \quad \text{as} \quad \mu\downarrow 0,$$

i.e.

$$c-U_1 \sim -De^{-j\pi/\mu} \text{ as } \mu\downarrow 0 \quad \text{for} \quad j = 1,2,\ldots, \tag{69}$$

where $D = U_1'e^{-A'/B'}$. Thus we identify $j = n-p$, noting that formulae (67) and (69) describe how all the modes approach U_1 from below.

The result embodied in (69) is again consistent with the numerical results obtained for the second mode. This was achieved by calculating $\pi^{-1}\mu\log(U_1-c)$ for smaller and smaller values of μ and thence confirming that this quantity was tending to -1 as required by (69).

In addition to the confirmation given above, the universal results (67) and (69) are exemplified by Høiland's (1953) work on plane Couette flow,

$$U = z, \quad N^2 = 1 \quad \text{for} \quad 0 \leqslant z \leqslant 1.$$

He found a solution which can be written as

$$\alpha^2 = 0, \quad c = -1/(e^{n\pi/\mu}-1) \quad \text{for} \quad n = 1,2,\ldots.$$

Now one can readily show that $p = 0$ for all α for this flow, so formula (69) is applicable with $U_1 = 0$; this is consistent with Høiland's result, which gives $D = 1$ for $\alpha = 0$. Further, for $\alpha = 1$, numerical calculations of Davey & Reid, who have kindly told us about their work on the stability of plane Couette flow of viscous stratified fluid in advance of publication, are also consistent with (69).

For the basic flow

$$U = 1-e^{-z}, \quad N^2 = e^{-z} \quad \text{for} \quad 0 \leqslant z < \infty,$$

Miles (1967) found the complete solution in terms of hypergeometric functions. His solution in our notation can be shown to give $p = 1$ for $\alpha < \frac{3}{4}$ with

$$c_1 \sim -\left\{\frac{(J-J_1)\,\Gamma(1+2\alpha)\,\Gamma(2\nu_1)}{2\nu_1\,\Gamma(1+2\alpha+2\nu_1)}\right\}^{1/2\nu_1} \quad \text{as} \quad J \downarrow J_1$$

and $\nu_1 = (1+\alpha^2)^{\frac{1}{2}} - \alpha - \frac{1}{2}$. For $\alpha > \frac{3}{4}$, $p = 0$. From his results it can also be deduced that

$$c \sim -\exp\left\{-\frac{1}{\nu_1} - \frac{j\pi}{\mu}\right\} \quad \text{as} \quad \mu \downarrow 0 \quad \text{for} \quad j = 1, 2, \ldots.$$

This agrees with (69), when $D = \exp(-1/\nu_1)$ and $j = n$ if $\alpha > \frac{3}{4}$ but $j = n-1$ if $\alpha < \frac{3}{4}$.

Finally we may look back over our asymptotic results and recognize that the arguments justifying (5), (27), (32) etc. seem to apply only when J/n^2 is large, that those justifying (43) apply when $\epsilon = 2J/n(n+2)$ rather than J is small, and that those justifying (69) apply when μ/j rather than μ is small. (Note that the matching of inner and outer solutions has depended essentially only upon the assumption that $c - U(z_c)$ is small, where $z = z_c$ is the position of the critical layer.) This suggests that our results for large J are invalid when J/n^2 is not large but that those for small J or μ are also valid when J/n^2 or μ/n is small although J or μ may not be small.

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